MATH 320 NOTES, WEEK 4

Section 1.6 Bases and Dimension cont'd
Recall:

Definition 1. *Suppose that $\beta \subset V$, for a vector space V. We say that β is a* **basis** *for V iff*

(1) *$Span(\beta) = V$,*
(2) *β is linearly independent.*

Next we want to define **the dimension** of a vector space V. The dimension of V will be the size of a basis for V. But for this notion to be well defined, we need two things:

(1) each vector space has a basis, and
(2) if β, γ are two bases for V, then they have the same size.

The next theorem addresses the first point.

Theorem 2. *Suppose that V is a vector space and $S \subset V$ is a finite set such that $span(S) = V$. Then V has a finite basis β with $\beta \subset S$.*

Proof. If $S = \emptyset$, then $V = \{\vec{0}\}$, so $\emptyset$ is a basis. So suppose that S is nonempty.

We prove the theorem by induction on $|S|$, i.e. the size of S. For the base case, if $|S| = 1$, then $S = \{v\}$, and we have two cases:

(1) if $v = \vec{0}$, then $V = \{0\}$ and $\emptyset$ is a basis.
(2) if $v \neq 0$, then $\{v\}$ is linearly independent and spans V. It follows that $\{v\}$ is a basis.

Now suppose that $|S| = n + 1$, for some $n > 0$ and the theorem is true for n: i.e. if a vector space W is such that $span(T) = W$ and $|T| = n$, then W has a basis contained in T.

Let $v \in S$ be nonzero (such a vector exists since S has at least two vectors). Let $T = S \setminus \{v\}$. Let $W = span(T)$. By the inductive hypothesis W has a basis $\beta \subset T$. We have two cases.

(1) if $v \in span(T)$, then $W = span(T) = span(T \cup \{v\}) = span(S) = V$ and β is a basis for V.
(2) if $v \notin span(T)$, then $v \notin span(\beta)$, and so by an earlier theorem $\beta \cup \{v\}$ is linearly independent. Let $\beta' = \beta \cup \{v\}$. Then β' is linearly independent. Also $span(\beta') = span(\beta \cup \{v\}) = span(T \cup \{v\}) = span(S) = V$. The second equality is since $span(\beta) = span(T)$. It follows that β' is a basis for V.

□

Example In $\mathbb{R}^2$, the set $S = \{(1,1),(1,2),(3,-3)\}$ generates $\mathbb{R}^2$, and $\{(1,1),(1,2)\}$ is a basis for $\mathbb{R}^2$.

Some vector spaces need an infinite set for a basis, for example $P(F)$. Using *Zorn's Lemma*, one can prove that every vector space has a basis (not only spaces generated by finite sets):

Theorem 3. *Let V be a vector space. Then there is a basis for V.*

Proof. (Informal) Look at the family of *all* linearly independent subsets L of V. Call this family $\mathcal{L}$. Using Zorn's Lemma, we can take a maximal element of $\mathcal{L}$ i.e. a linearly independent set β, such that for every vector $x \in V$, if $x \notin \beta$, then $\beta \subset \{x\}$ is not linearly independent. Then β is a basis. The hardest part of the proof is to show that $\mathcal{L}$ satisfies the requirements of Zorn's lemma. □

Theorem 4. *(Replacement theorem) Let V be a vector space, and $G, L \subset V$ be such that:*

(1) *$span(G) = V$ and $|G| = n$,*
(2) *L is linearly independent, and $|L| = m$.*

Then $m \leq n$, and there is $H \subset G$ with $|H| = n-m$, such that $span(L \cup H) = V$.

Proof. By induction on m.

Base case: $m = 0$. Then clearly $m \leq n$ and $n - m = n$, so we can take $H = G$.

Inductive case: suppose that $L = m + 1$ and the theorem is true for m. I.e. the inductive hypothesis is that if L' is a linearly independent set of size m, then $m \leq n$ and there is $H' \subset G$ such that $H' \cup L'$ generates V and $|H'| = n - m$.

Let $v \in L$, and set $L' = L \setminus \{v\}$. Then L' is linearly independent and $|L'| = m$, and so by the inductive hypothesis, $m \leq n$ and let $H' \subset G$ be such that $H' \cup L'$ generates V and $|H'| = n - m$.

First we will show that $m + 1 \leq n$. Since L is linearly independent, we know that $v \notin span(L')$. On the other hand $v \in span(L' \cup H') = V$. It follows that H' is nonempty, i.e. that $n - m > 0$. So $m + 1 \leq n$.

Next, we have to *replace* one of the vectors in H' with v.

Denote $H' = \{u_1, ..., u_{n-m}\}$ and $L' = \{v_1, ..., v_m\}$. Since $v \in span(H' \cup L')$, there are scalars $a_1, ..., a_m, b_1, ..., b_{n-m}$ such that

$$v = a_1 v_1 + a_2 v_2 + ... + a_m v_m + b_1 u_1 + ... + b_{n-m} u_{n-m}.$$

Since $v \notin span(L')$, it must be that at least one of the b_i's is nonzero. By rearranging the u's if necessary, we may assume that $b_1 \neq 0$. Then from the above equation we can solve for u_1. In particular,

$$b_1 u_1 = v - a_1 v_1 - a_2 v_2 + ... - a_m v_m - b_2 u_2 - ... - b_{n-m} u_{n-m},$$

so

$$u_1 = \frac{1}{b_1} v - \frac{a_1}{b_1} v_1 - ... - \frac{a_m}{b_1} v_m - \frac{b_2}{b_1} u_2 + ... - \frac{b_{n-m}}{b_1} u_{n-m}.$$

Hence, $u_1 \in span(L \cup \{u_2, ..., u_{n-m}\})$.

Set $H = \{u_2, ..., u_{n-m}\}$. Then $|H| = n - m - 1$, and $span(L \cup H) = span(L \cup \{u_1, u_2, ..., u_{n-m}\}) = span(L \cup H')$. Then last equality is by the fact that $u_1 \in span(L \cup H)$.

It follows that $span(L \cup H) = span(L \cup H') \supset span(L' \cup H') = V$. So $span(L \cup H) = V$.

□

As a corollary, we can show that any two finite bases for the same vector space have the same number of vectors.

Corollary 5. *Suppose that V is a vector space and β, γ are two finite bases for V. Then $|\beta| = |\gamma|$.*

Proof. Apply the Replacement theorem for $G = \beta$ and $L = \gamma$, to get that $|\beta| = |G| \geq |L| = |\gamma|$.

Next, by the Replacement theorem applied for $G = \gamma$ and $L = \beta$, we get that $|\gamma| = |G| \geq |L| = |\beta|$.

So $|\beta| = |\gamma|$. □

Remark 6. It also true that any two infinite bases have the same size. For example, any basis for $P(F)$ is *countable*, i.e. you can put it in a one-to-one correspondence with the natural numbers. On the other hand any basis for the vector space of functions $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is uncountable.

Here we omit these proofs.

So, now we have that the definition of dimension is well-defined:

$$\dim(V) = |\beta|, \text{ where } \beta \text{ is a basis for } V.$$

We say that V is **finite dimensional** if $\dim(V)$ is finite. Otherwise V is infinite dimensional.

Examples:

(1) $\mathbb{R}^3$ over $\mathbb{R}$ has dimension 3;
(2) F^n over F has dimension n;
(3) $M_{k,n}(F)$ over F has dimension kn;
(4) $P_n(F)$ over F has dimension $n + 1$;
(5) $P(F)$ is infinite dimensional.
(6) $\mathcal{F}(\mathbb{R}, \mathbb{R})$ is infinite dimensional;
(7) $\mathbb{R}^3$ over $\mathbb{Q}$ is infinite dimensional – so note that the choice of field matters.

And now for the other payoff of the replacement theorem:

Corollary 7. *Suppose that V is vector space and* $\dim(V) = n$. *Then*

(1) *Any generating set contains a basis.*
(2) *If β spans V and $|\beta| = n$, then β is a basis.*
(3) *Any linearly independent set L can be extended to a basis.*
(4) *If β is linearly independent and $|\beta| = n$, then L is a basis.*

Proof. We already proved (1) in an earlier theorem.

(2). By (1), there is $\alpha \subset \beta$, such that α is a basis for V. Since $\dim(V) = n$, $|\alpha| = n$. But then $\alpha = \beta$ (since they have the same finite size and $\alpha \subset \beta$).

(3). Suppose that L is linearly independent. Let α be a basis for V. Then $|\alpha| = n$ and $|L| = m \leq n$. By the Replacement theorem applied to L and $G = \alpha$, there is a set $H \subset \alpha$, such that $span(L \cup H) = V$, and $|L \cup H| = n$. By (2), $L \cup H$ is a basis for V.

(4). Suppose that β is linearly independent of size n. By (3), β can be extended to a basis α for V. But then $|\alpha| = |\beta| = n$ and $\beta \subset \alpha$. Then $\beta = \alpha$.

□

Let us summarize: If V is a vector space with dimension n and $\beta \subset V$ has size n. Then TFAE:

(1) β is a basis;
(2) β is linearly independent;
(3) $span(\beta) = V$.

That means that when you prove that a set is a basis, after checking that it has the right number of vectors, it is enough to prove that it is linearly independent, or to prove that it generates the vector space.